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ECONOMIA — ECONOMICS

Nº 04/2008

A COMMUNICATION EQUILIBRIUM IN ENGLISH AUCTIONS WITH DISCRETE BIDDING

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A Communication Equilibrium in English Auctions with Discrete Bidding*

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June 2008

Abstract

This paper analyses a model of a common value English auction with discrete bidding. In this model, we show that there exists a communication equilibrium in which the high signal bidder strategically chooses his first bid so as to maximise his expected utility. Straightforward bidding, or increasing the bid by the minimum amount possible, is the equilibrium strategy for both bidders in all other auction rounds. We relate this result to recent research on English auctions with discrete bidding and auctions where bidders may have noisy information about their opponent's signals.

JEL Classification: D44

Keywords: English Auctions, discrete bidding, communication equilibrium.

*This paper is based on Chapter 2 of my PhD thesis from the University of York. I wish to thank John Bone, John Hey, Norman Ireland, Indrajit Ray and especially Peter Simmons for many useful comments and suggestions. Financial support from Foundation for Science and Technology (Portugal) is gratefully acknowledged.

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1 Introduction

English auctions are probably the most popular type of auction. Cassady (1967) suggests that more than 75% of all auctions are of the English type. Despite its popularity, it is rather complicated to model a real world English auction, as acknowledged by Matthews (1995). Milgrom and Weber (1982) and Bikhchandani and Riley (1991) modeled a particular version of the English auction, the so called Japanese English auction: the price increases continuously and interested bidders must depress a button as long as they are interested in the good for sale. When all but one bidder release the button, the auction finishes. The price, the number of bidders and the drop-out prices of all bidders are displayed for all to see. Kamecke (1998) suggests that modeling an English auction in this way eliminates many of the interesting features which probably make it so popular.

Daniel and Hirshleifer (1998), Avery (1998) and Kamecke (1998) built models where more dynamic aspects of the single unit English auction were captured. Avery (1998) solves an affiliated values[1] English auction model and shows that jump bidding is an optimal strategy for high signal bidders and that it is more likely to occur early in the auction. Daniel and Hirshleifer (1998) obtain a similar jump bidding equilibrium in a setup where bidding is costly. Kamecke (1998) shows that even in a private values model without bidding costs, both straightforward bidding (to increase the current bid by as little as possible) and the jump bidding equilibrium remain in the equilibrium set even after the iterated dominance criterion is applied, and that more restrictions are necessary to ensure that the straightforward bidding equilibrium is unique.

We also think that the Japanese English auction model misses out some interesting features of more "traditional" English auctions: bid levels are typically discrete[2]; they are not always known by the bidders[3]; bidders can choose what bid to make at each stage (endogenous bidding)[4]; and the bidding is open to *all* bidders until the auction terminates[5]. Online auction sites, such as eBay, Yahoo, Amazon or QXL, use variants of such "traditional" English auctions, adapted to the online world (see Bajari and Hortaçsu (2004), Ockenfels et al. (2006) or Lucking-Reiley (2000)).

With some exceptions (e.g. Rothkopf and Harstad (1994), Sinha and Greenleaf (2000), Cheng (2004), Cox (2005), David et al. (2005) and Isaac et al. (2007)), most of the literature has considered the Japanese English auction to be a good representation of an English auction. Even

[1] Avery's (1998) model nests the independent private values and the common value models as special cases.

[2] In auctions at Sotheby's or Christie's, bidding usually advances between 5% and 10% of the current price level (Rothkopf and Harstad (1994) and Sinha and Greenleaf (2000)). In Internet auctions, the auction sites usually restrict bid levels to be integers which vary according to the current price level (e.g. auctions at eBay (http://www.ebay.com) or QXL (http://www.qxl.com)).

[3] Cassady (1967) gives examples of auctions in which the bid levels are known (tobacco and livestock auctions in the USA) and unknown (some antique auctions in London).

[4] In auctions where the bid levels are known, a bidder knows (when making a bid) what is the next bid the auctioneer will call for. By contrast, in takeover bids, there is no auctioneer calling for bids; bidders must call their own bids.

[5] In Milgrom and Weber's (1982) model, once a bidder drops out, he is not allowed back in the bidding.

jump bidding[6] was analyzed by Avery (1998) and Kamecke (1998) using a model which is based on the Japanese English auction. More recently, Isaac et al. (2007) put forward a theory to explain jump bidding in independent private value English auctions with discrete bidding. Their model allows for jump bidding because of (i) strategic concerns, i.e. placing a certain bid to prevent opponents' from doing so, or (ii) impatience, i.e. bidder's preference for finishing the auction with fewer bids. An experimental test of this theory, carried out by Isaac et al. (2005), suggests that both factors may help explain the prevalence of jump bidding in English auctions with discrete bidding.

We propose an English auction model which allows for the price to be raised in discrete levels known to the bidders prior to the auction start. One basic assumption regarding the discreteness of bids divides the literature into two groups. On the one hand, some authors assume the English auction has discrete predefined *bid levels*, and bidders have to choose among those bid levels when it is their turn to bid (Rothkopf and Harstad (1994) and David et al. (2005)). English auctions in the online auction site QXL and eBay work in this way, although the latter has a proxy bidding mechanism which makes it substantially different from auctions at QXL. On the other hand, other authors model the English auction has having discrete predefined *bid increments* and bidders, when considering placing a bid, know that they have to raise the bid, at the very least, by that increment (Cox (2005) and Isaac et al. (2005)). In the online football game Hattrick, players can be bought at an English auction where the bid must always increase by at least €1000 or by 2% of the current bid, whichever is highest. Similarly, in the UK's ascending bid auction for 3G mobile licenses in 2000, the price would have to increase by at least 5% of the current bid[7]; in the Broadband Fixed Wireless Spectrum Auction (BFWA) the minimum increment was set at 15% of the current bid[8]. Our model is based on the first assumption - known predefined bid levels - and, in fact, it borrows many similarities to the way auctions are conducted at the online auction site QXL.

Additionally, and differently from the previous literature, ours is a common value model in which the value of the good is identical to both bidders but *ex ante* unknown. This appears to be a more plausible assumption for some auctions, e.g. spectrum auctions, where jump bidding has been observed and extensively analyzed (Cramton (1997), Börgers and Dustmann (2005) or Plott and Salmon (2004)), or online auctions where bidders participate with the objective of reselling the good.

We find a communication equilibrium (Forges (1986) and Myerson (1986)) whereby the presence of an abstract device or mediator may coordinate the bidder's strategies in an incentive compatible

[6] Which became popular in the recent FCC auctions (see Cramton (1997)), although Avery (1998) notes it was used with great success in art auctions in the 1930's, as well as in takeover bids.

[7] See paragraph 4.3.17 of the auctions rules available at http://www.ofcom.org.uk/static/archive/spectrumauctions/3gindex.htm.

[8] Paragraph 4.3.22 of the auction rules, available at http://www.ofcom.org.uk/static/archive/ra/topics/bfwa/doc28ghz/auc_not/notice.htm.

way. In this communication equilibrium, the choice of the starting bid is strategic (in a way similar to Isaac et al. (2007)), as high signal bidders choose the starting bid which maximizes their expected utility. After the initial auction stage, both bidders bid straightforwardly, i.e. they increase their bid by the minimum amount possible, until their bidding limits are reached, and these bidding limits are similar to those obtained by Milgrom and Weber (1982) or Klemperer (1998). This equilibrium combines Rothkopf and Harstad's (1994) result that straightforward bidding is an equilibrium in private values English auctions with discrete bidding with Isaac et al.'s (2007) result that bidders may strategically choose the bidding path which favors them the most.

This communication equilibrium can be played without the need for a mediator if the signal ranking is common knowledge, i.e. if, before the auction starts, both bidders know who holds the highest signal (but not the particular signal realizations). This is not a common assumption in auction models, but recent literature has analyzed the impact of such departures from the standard assumptions. Fang and Morris (2006) assume that each bidder in an independent private values auction observes his private valuation as well as a noisy and private signal about their opponent's valuation. Kim and Che (2004) assume that subgroups of bidders perfectly observe their own valuations, but not those of other groups, in an independent private value auction. Kim (2007) proposes a further extension where each bidder's noisy signals about their opponent's valuations are common knowledge. Jofre-Bonet and Pesendorfer (2003) suggest that in highway construction procurement auctions, a bidder's capacity utilization can be a determinant of their costs; Fang and Morris (2006) note that rival firms may thus try to infer a bidder's costs based on their capacity utilization levels. As we will explain, if the signal ranking becomes common knowledge, we are effectively introducing into the auction model a particular form of Fang and Morris' (2006) private and noisy signals.

The paper is organized in the following way: the next section introduces some definitions, section 3 contains the main results, section 4 presents an illustrative example and section 5 concludes.

2 The model

There are two symmetric risk-neutral bidders $i \in \{1, 2\}$ who compete for the purchase of one single good, whose value, V, is common but *ex ante* unknown to both bidders. Each bidder receives a signal $X_i \in [\underline{x}, \overline{x}]$ of the common value and the signals are identically distributed. We assume that the common value of the good depends on the two signals: $V = V(x_1, x_2) = V(x_2, x_1)$, where $V(.)$ is continuous and strictly increasing in each of its arguments. We also assume that the signals are affiliated, which means that a high signal for one of the bidders makes the other signal being high more likely rather than less likely (for more details see Milgrom and Weber (1982) or Bikhchandani and Riley (1991)). This assumption requires that for all $x_i' > x_i$, and $x_j' > x_j$:

$$\frac{g_{X_i|X_j}\left(x_i|\, x_j\right)}{g_{X_i|X_j}\left(x_i|\, x_j'\right)} \geq \frac{g_{X_i|X_j}\left(x_i'|\, x_j\right)}{g_{X_i|X_j}\left(x_i'|\, x_j'\right)} \tag{1}$$

i.e. $g_{X_i|X_j}\left(x_i|\, x_j\right)$, the conditional signal distribution, must satisfy the Monotone Likelihood Ratio Property (see Milgrom and Weber (1982)).

Bidding is alternate, so that each bidder is only allowed to submit a bid when it is his turn to bid, and that bid must be higher or equal than the minimum allowed bid at that stage. We assume the auctioneer sets discrete bid levels $A = \{a_0, a_1, ..., a_L\}$, with $a_L > ... > a_1 > a_0$, L finite and $a_L \geq V(\overline{x}, \overline{x})$, which are common knowledge to bidders. However, whenever it is their turn to bid, bidders are freely allowed to submit any bid above the minimum allowed bid and this will have an impact on the minimum allowed bid their opponent faces in the next bidding round: the lowest bid level in A above their bid. Formally, if at a given bidding round, the minimum allowed bid is $a_k \in A$, the active bidder can bid any amount in the interval $[a_n, a_{n+1})$ and this will cause the minimum allowed bid in the subsequent stage to be a_{n+1}, where $n = k, ..., L-1$. Rothkopf and Harstad (1994) and David et al. (2005) assume a similar setup but restrict bidders to submit bids contained in A. An online auction site, QXL[9], has similar bidding rules to those of our model: the price goes up in predetermined increments[10] and if bids are not a multiple of that increment, then the bid is rounded down to the closest multiple of the increment (in our setup, this rounding down does not occur but the impact on the minimum allowed bid in the subsequent stage is similar).

Milgrom and Weber (1982) analyzed the equilibrium bidding strategies in a particular version of the English auction - the Japanese English auction - where the price increases continuously and interested bidders must depress a button as long as they are interested in the good; the auction finishes when the penultimate bidder releases the button (i.e. when only one bidder remains interested at a certain price level). In their model, Milgrom and Weber (1982) show that the equilibrium bidding strategy for bidder i is to remain active in the auction (depressing the button) until the price reaches $V(x_i, x_i)$, i.e. $V(x_i, x_i)$ would be bidder i's bidding limit. This is the only symmetric equilibrium of this auction (Levin and Harstad (1986)).

It is simple to show that this would indeed be an equilibrium strategy (Klemperer (1998)). Suppose bidder 1's strategy is to remain active in the auction until the price reaches $V(x_1, x_1)$. If bidder 2 holds the highest signal (i.e. if $x_1 < x_2$), remaining active until the prices reaches $V(x_2, x_2)$ is a best reply: no other strategy yields a better result than winning the auction and obtaining a positive payoff equal to $V(x_1, x_2) - V(x_1, x_1) > 0$ (the ex post value of the good minus the price bidder 2 has to pay for it); on the other hand, if bidder 2 holds the lowest signal (i.e. if $x_1 > x_2$) then remaining active until the price reaches $V(x_2, x_2)$ is also a best reply: no other

[9]The UK site of QXL, www.qxl.co.uk, closed down in May 2008, but the bidding rules are still available for consultation.

[10]These increments depend on the bid value. For example, for bids in the £2.50-£9.99 range, the bid increment is £0.10, for bids in the £10-£99.99 it is £1.00 and for bids in the £100-£499 it is £5.00.

strategy yields a better result than not winning the auction and obtaining a payoff equal to zero (if bidder 2 deviated from this bidding limit and continued bidding, he would risk winning the auction and obtaining a negative payoff equal to $V(x_1, x_2) - V(x_1, x_1) < 0$).

As we will see, these bidding limits also hold for the equilibrium strategies in our model. Therefore, it is useful to define B_i, a subset of A for bidder i, as:

$$B_i = \{ a_k \in A | \, a_k \leq V(x_i, x_i) \}, i = 1, 2 \tag{2}$$

Hence, the set B_i contains all the bid levels which are below $V(x_i, x_i)$. We define the maximum of set B_i to be $s_i^H = \max(B_i)$,i.e. s_i^H is the highest bid level (hence the superscript H) contained in B_i. This implies that if bidder i were to bid s_i^H at some point in the auction, then the minimum allowed bid in the subsequent round would no longer belong to B_i, i.e. it would exceed $V(x_i, x_i)$. Also define $s_i^{SH} = \max(B_i \backslash \{s_i^H\})$, i.e. s_i^{SH} is the second highest bid level (hence the superscript SH) contained in B_i. Finally, let $n_i = n(B_i)$ be the number of bid levels contained in B_i[11]

Figure 1 contains a brief graphical description of these definitions for a bidder 2 who receives a signal x_2 such that $a_4 < V(x_2, x_2) < a_5$. In this case, $B_2 = \{a_0, a_1, a_2, a_3, a_4\}$, $s_2^H = a_4$, $s_2^{SH} = a_3$ and $n_2 = 5$ (the number of elements of B_2). Figure 1 also contains a description of the possible bidding strategies available to bidder 2. Suppose both bidders always increase their bids by the least amount possible (a strategy defined by Rothkopf and Harstad (1994) as "pedestrian bidding" and by Isaac et al. (2007) as "straightforward bidding" - we will use the latter term throughout this paper). In this case, if bidder 2 chooses $V(x_2, x_2)$ to be his bidding limit, then two options are available to him: he can choose a bidding path such that it is him who bids $s_2^H = a_4$ or he can choose a bidding path such that it is him who bids $s_2^{SH} = a_3$. In both cases, those would be his final expected bids: if bidder 2's opponent were to bid higher than either $s_2^{SH} = a_3$ or $s_2^H = a_4$, then the next available bid level for bidder 2 would be above his bidding limit, $V(x_2, x_2)$.

The choice between these two bidding paths has implications at the start of the auction. In this example, if bidder 2 were to choose the bidding path such that it is him who bids $s_2^H = a_4$, then he would have to start the auction (assuming both bidders engage in straightforward bidding). By contrast, if bidder 2 were to choose the bidding path such that it was him who bid $s_2^{SH} = a_3$, then he would have to start the auction with a bid of a_1 (larger than the minimum possible bid at that stage, a_0), or let bidder 1 start the auction, and engage in straightforward bidding subsequently. We define these two strategies as the "closest strategy", when he chooses the bidding path which leads to a final bid (of all the bid levels contained in A) as close as possible to (but lower than) $V(x_2, x_2)$, and the "second closest strategy", when he chooses the bidding path leading to a final bid which is the second closest bid level to (but lower than) $V(x_2, x_2)$ contained in A. Note that, in this example, choosing to play the "closest strategy" is equivalent to straightforward bidding

[11] We use the notation $n(F)$ to represent the number of elements in set F.

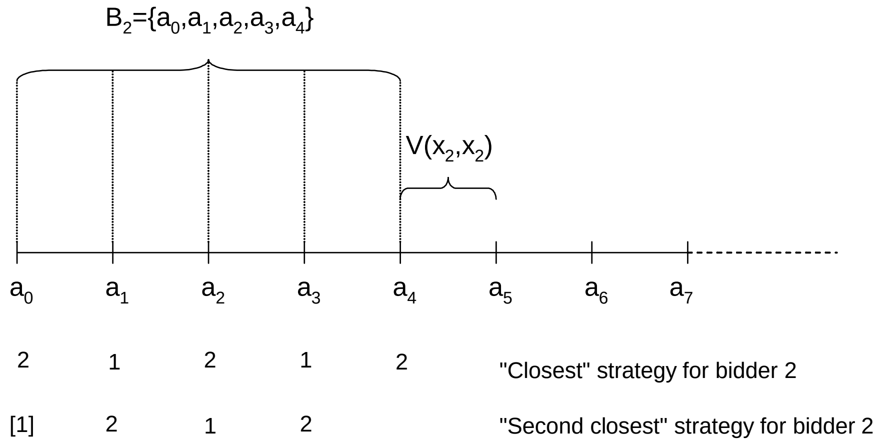

Figure 1: Bid levels and bidding strategies for bidder 2

up to the bidding limit. By contrast, choosing to play the "second closest strategy" is equivalent to straightforward bidding up to the bidding limit with the exception of the first bidding round (where bidder 2 would have to bid more than the minimum possible bid, a_0, or let bidder 1 start the auction).

Definition 1 *A bidder i plays the "closest strategy" when he chooses a bidding path such that (i) the last bid he plans to place is equal to s_i^H and (ii) in all but the first auction round he bids exactly the minimum allowed bid at that stage (straightforward bidding). Because bidding is alternate, in order to be able to place a final bid of s_i^H, he may have to start the auction with a bid of a_0 or a_1.*

Definition 2 *A bidder i plays the "second closest strategy" when he chooses a bidding path such that (i) the last bid he plans to place is equal to s_i^{SH} and (ii) in all but the first auction round he bids exactly the minimum allowed bid at that stage (straightforward bidding). Because bidding is alternate, in order to be able to place a final bid of s_i^{SH}, he may have to start the auction with a bid of a_0 or a_1.*

Under these assumptions, it is possible to say that when B_i contains an odd number of elements, i.e. n_i is odd, then playing the "closest strategy" implies that bidder i must start the auction with a bid of a_0 and playing the "second closest strategy" implies that bidder i must start the auction with a bid of a_1 (as shown in the above example). By contrast, if n_i is even, the reverse holds: playing the "closest strategy" implies that bidder i must start the auction with a bid of a_1 and playing the "second closest strategy" implies that bidder i must start the auction with a bid of a_0.

Consider bidder i who receives a signal x_i. Define:

$$z\left(a_k, a_{k+1}\right) = G_{X_j|X_i}\left(V^{-1}\left(a_{k+1}\right)\right) - G_{X_j|X_i}\left(V^{-1}\left(a_k\right)\right) \tag{3}$$

i.e. $z(a_k, a_{k+1})$ is the probability that $a_k \leq V(x_j, x_j) < a_{k+1}$ given that bidder i's signal is x_i. In other words, $z(a_k, a_{k+1})$ is the probability that bidder i's opponent (bidder j) bidding limit, $V(x_j, x_j)$, belongs to the interval $[a_k, a_{k+1})$. Also define:

$$V_{a_k,a_{k+1}} = E\left[V(X_1, X_2) \mid a_k \leq V(x_j, x_j) < a_{k+1}, X_i = x_i\right] \tag{4}$$

This is the expected value of the good if bidder i expects bidder j's bidding limit, $V(x_j, x_j)$, to belong to the interval $[a_k, a_{k+1})$.

Let us assume that there are no ties, which means that the bidding limits of the two bidders do not fall in the same bidding interval: for this to happen, it must be that the signal realizations are such that $V(x_1, x_1) \in [a_b, a_{b+1})$ and $V(x_2, x_2) \in [a_c, a_{c+1})$, with $b \neq c$. In this case, the expected utility of the "closest strategy" (CS) for bidder i is given by:

$$\begin{aligned} U_i^{CS}(x_i) \;=\; & \left[\sum_{\forall k \in (t, t+2, t+4, \dots, n_i-5, n_i-3)} z(a_{k-1}, a_k)\left(V_{a_{k-1},a_k} - a_k\right) + z(a_k, a_{k+1})\left(V_{a_k,a_{k+1}} - a_k\right)\right] + \\ & + z(a_{n_i-2}, a_{n_i-1})\left(V_{a_{n_i-2},a_{n_i-1}} - a_{n_i-1}\right) + z(a_{t-2}, a_{t-1})\left(V_{a_{t-2},a_{t-1}} - a_{t-2}\right) \end{aligned} \tag{5}$$

where $t = 2$ if n_i is odd and $t = 1$ if n_i is even. Using the above example, this expression would yield:

$$\begin{aligned} U_2^{CS}(x_2) = & \left[z(a_1, a_2)\left(V_{a_1,a_2} - a_2\right) + z(a_2, a_3)\left(V_{a_2,a_3} - a_2\right)\right] + \\ & + z(a_3, a_4)\left(V_{a_3,a_4} - a_4\right) + z(a_0, a_1)\left(V_{a_0,a_1} - a_0\right) \end{aligned} \tag{6}$$

The expected utility of the "closest strategy" is the expected surplus of bidder i whenever he places a bid prescribed by the "closest strategy", multiplied by the respective probability of winning (assuming his opponent engages in straightforward bidding). Similarly, the expected utility of the "second closest strategy" (SCS) for bidder i is given by:

$$\begin{aligned} U_i^{SCS}(x_i) \;=\; & \left[\sum_{\forall k \in (t, t+2, t+4, \dots, n_i-4, n_i-2)} z(a_{k-1}, a_k)\left(V_{a_{k-1},a_k} - a_k\right) + z(a_k, a_{k+1})\left(V_{a_k,a_{k+1}} - a_k\right)\right] + \\ & + z(a_{t-2}, a_{t-1})\left(V_{a_{t-2},a_{t-1}} - a_{t-2}\right) \end{aligned} \tag{7}$$

where $t = 1$ if n_i is odd and $t = 2$ if n_i is even. Using the above example, this expression would yield:

$$\begin{aligned} U_2^{SCS}(x_2) \;=\; & z(a_0, a_1)\left(V_{a_0,a_1} - a_1\right) + z(a_1, a_2)\left(V_{a_1,a_2} - a_1\right) + \\ & + z(a_2, a_3)\left(V_{a_2,a_3} - a_3\right) + z(a_3, a_4)\left(V_{a_3,a_4} - a_3\right) \end{aligned} \tag{8}$$

as $t = 1$ in this case and the the last element in equation (7) is equal to 0 because $a_{t-2} = a_{-1} \notin A$ and hence $z(a_{-1}, a_0) = 0$. The expected utility of the "second closest strategy" is the expected surplus of bidder i whenever he places a bid prescribed by the "second closest strategy", multiplied by the respective probability of winning (assuming his opponent engages in straightforward bidding).

The difference between the expected utility of the "closest strategy" and the "second closest strategy" is given by:

$$U_i^{CS}(x_i) - U_i^{SCS}(x_i) = \sum_{\forall k \in (t, t+2, \ldots, n_i - 4, n_i - 2)} z(a_{k-1}, a_k)(a_k - a_{k-1}) + z(a_k, a_{k+1})(a_k - a_{k+1}) \quad (9)$$

where $t = 1$ if n_i is odd and $t = 0$ if n_i is even. Using the above example, this expression would yield:

$$\begin{aligned} U_2^{CS}(x_2) - U_2^{SCS}(x_2) &= z(a_0, a_1)(a_1 - a_0) + z(a_1, a_2)(a_1 - a_2) + \\ &\quad + z(a_2, a_3)(a_3 - a_2) + z(a_3, a_4)(a_3 - a_4) \end{aligned} \quad (10)$$

Naturally, when $U_i^{CS}(x_i) - U_i^{SCS}(x_i) \geq 0$, bidder i prefers the "closest strategy". This preference is essentially the choice of the bidding path which favors that bidder the most. In general, there are three factors which may influence this choice: (i) the signal distribution (which affects $z(a_{k-1}, a_k)$), (ii) the predefined bid levels (set A) and (iii) the particular signal realization (because the signal distribution is affiliated). In the above example, if the "closest strategy" were preferred to the "second closest strategy", this would imply that the bidding path in which bidder 2 starts the auction with bid of a_0 and submits a final bid of a_4 results in a higher expected utility than the path in which he starts the auction with a bid of a_1 and submits a final bid of a_3.

3 The communication equilibrium

The equilibrium concept we will make use of is the extensive form definition of correlated equilibrium, given by Forges (1986) and Myerson (1986). The notion of correlated equilibrium, introduced by Aumann (1974) and further strengthened in Aumann (1987) for games in the normal form, is extremely useful, for it generalizes the existence of a mixed strategy equilibrium for any finite game. A correlated equilibrium is basically a mixed strategy equilibrium in which the probabilities attached to each pure strategy need not be independent across players. These correlated probabilities can support an equilibrium if there exists a mediator or device through which preplay communication is allowed.

For games in the extensive form in which there could be not only preplay communication but also intraplay communication, the definition we will make use of is that of a *communication equilibrium* (see Forges (1986) or Myerson (1986) for more details). A device or mediator will be added to the extensive form definition of the game, and this device will receive inputs from the players (e.g. information signals), compute a recommendation based on all the information collected, recommend to each player a particular action and observe their decision. A communication equilibrium will be obtained if every player reveals his private information truthfully to the device and obeys its recommendation. Remember that the revelation principle asserts that any Nash equilibrium of the game obtained by any other communication mechanism (direct or indirect) can be mimicked by a communication equilibrium with truthful revelation of private information and obedience to the mechanism's recommendations. Hence, restricting our attention to incentive compatible mechanisms will not affect the set of Nash equilibria of the auction game (see Myerson (1986)).

Thus, we will add a mediator to our auction game, to whom bidders will have to report their signal realizations: define x_i^R to be the signal bidder i reports to the mediator. Also define n_i^R to be the number of elements of set B_i given a signal of x_i^R. The mediator will then recommend a bidding strategy to each bidder. As we have observed above, in a communication equilibrium both bidders will report their signals truthfully to the mediator and obey his recommendations.

Proposition 1 *Assuming there are no ties, there exists a communication equilibrium in which the mediator recommends the high signal bidder to start the auction, i.e. if $x_i^R > x_j^R$ the mediator recommends bidder i to start the auction. When n_i^R is odd and $U_i^{CS}\left(x_i^R\right) - U_i^{SCS}\left(x_i^R\right) \geq 0$, the mediator recommends bidder i to start the auction with a bid of a_0 and from then onwards to bid the minimum allowed bid up to s_i^H (closest strategy); conversely, when n_i^R is odd and $U_i^{CS}\left(x_i^R\right) - U_i^{SCS}\left(x_i^R\right) < 0$, the mediator recommends bidder i to start the auction with a bid of a_1 and from then onwards to bid the minimum allowed bid up to s_i^{SH} (second closest strategy). When n_i^R is even and $U_i^{CS}\left(x_i^R\right) - U_i^{SCS}\left(x_i^R\right) \geq 0$, the mediator recommends bidder i to start the auction with a bid of a_1 and from then onwards to bid the minimum allowed bid up to s_i^H (closest strategy); conversely, when n_i^R is even and $U_i^{CS}\left(x_i^R\right) - U_i^{SCS}\left(x_i^R\right) < 0$, the mediator recommends bidder i to start the auction with a bid of a_0 and from then onwards to bid the minimum allowed bid up to s_i^{SH} (second closest strategy). The mediator never recommends the low signal bidder, bidder j, to start the auction and always recommends him to bid the minimum allowed bid up to s_j^H (straightforward bidding).*

Proof. The communication mechanism will be a communication equilibrium if both bidders report their signals truthfully and obey the mediator throughout the auction game. We first assume they report truthfully and show that obedience is guaranteed. We then prove that they always report truthfully. For expositional purposes, we assume that bidder 2 holds the highest signal $x_2 > x_1$.

(i) Obedience to the mediator

(Bidder 2) Assuming both bidders reported their signals truthfully, bidder 2 is the high signal bidder. Hence, the mediator recommends straightforward bidding to bidder 2 after the first auction stage. Straightforward bidding is an equilibrium strategy because given that bidder 1 is also bidding straightforwardly, increasing the bid above the minimum allowed bid at any auction stage (other than the first one) would not increase bidder 2's probability of winning the auction if the bid was still below bidder 1's bidding limit (i.e. such a strategy would be weakly dominated) and would reduce bidder 2's payoff if the bid was above bidder 1's bidding limit (i.e. such a strategy would be strictly dominated), because that would be the price bidder 2 had to pay.

Assuming truthful reports by both bidders, and as we have seen in equation (9), bidder 2 prefers to play the closest strategy (and start the auction with a_0) when n_2 is odd and $U_2^{CS}(x_2) - U_2^{SCS}(x_2) \geq 0$, but would rather play the second closest strategy (and start the auction with a_1) when $U_2^{CS}(x_2) - U_2^{SCS}(x_2) < 0$. A similar reasoning applies to the case where n_i is even. This suggests that bidder 2 would obey the mediator's recommendation.

However, by being recommended to start the auction, bidder 2 also learns that he is the high signal bidder: he now knows that $X_1 \leq x_2$. Hence, the density function bidder 2 should use to calculate $U_2^{CS}(x_2) - U_2^{SCS}(x_2)$ now becomes:

$$g'_{X_1|X_2}(x|\, X_2 = x_2; X_1 \leq x_2) = \begin{cases} \frac{g_{X_1|X_2}(x|X_2=x_2)}{G_{X_1|X_2}(x_2|X_2=x_2)}, \text{ if } 0 \leq x \leq x_2 \\ 0, \text{ if } x_1 \leq 0 \text{ or } x_1 \geq x_2 \end{cases} \tag{11}$$

and the respective distribution function should be:

$$G'_{X_1|X_2}(x|\, X_2 = x_2; X_1 \leq x_2) = \begin{cases} 0, \text{ if } x \leq 0 \\ \frac{\int_0^x g_{X_1|X_2}(x|X_2=x_2)dx_1}{G_{X_1|X_2}(x_2|X_2=x_2)}, \text{ if } 0 \leq x \leq x_2 \\ 1, \text{ if } x \geq x_2 \end{cases} \tag{12}$$

These new truncated density and distribution functions take this form because:

$$G'_{X_1|X_2}(x|\, X_2 = x_2; X_1 \leq x_2) = \frac{\Pr(X_1 \leq x; X_2 = x_2; X_1 \leq x_2)}{\Pr(X_1 \leq x_2; X_2 = x_2)} = \frac{\Pr(X_1 \leq x; X_2 = x_2)}{\Pr(X_1 \leq x_2; X_2 = x_2)} \tag{13}$$

as the realization of the event $X_1 \leq x_2$ necessarily causes the event $X_1 \leq x$ (when $x \leq x_2$) to occur. This implies that equation (3) now becomes:

$$\begin{aligned} z'(a_k, a_{k+1}) &= G'_{X_1|X_2}\left(V^{-1}(a_{k+1})\right) - G'_{X_1|X_2}\left(V^{-1}(a_k)\right) \\ &= \frac{1}{G_{X_1|X_2}(x|\, X_2 = x_2)} z(a_k, a_{k+1}), \ \forall a_k \in B_2 \end{aligned} \tag{14}$$

and equation (4) becomes:

$$\begin{aligned} V'_{a_k,a_{k+1}} &= E\left[V(X_1, X_2)|\, a_k \leq V(x_1, x_1) < a_{k+1}; X_2 = x_2; X_1 \leq x_2\right] \\ &= E\left[V(X_1, X_2)|\, a_k \leq V(x_1, x_1) < a_{k+1}; X_2 = x_2\right] \\ &= V_{a_k,a_{k+1}}, \ \forall a_k \in B_2 \end{aligned} \tag{15}$$

Therefore, equation (9) now becomes:

$$U_2'^{CS}(x_2) - U_2'^{SCS}(x_2) = \frac{1}{G_{X_1|X_2}(x|\, X_2 = x_2)} \left[U_2^{CS}(x_2) - U_2^{SCS}(x_2)\right] \tag{16}$$

Hence, if $U_2^{CS}(x_2) - U_2^{SCS}(x_2) \geq 0$, then it is necessarily true that $U_2'^{CS}(x_2) - U_2'^{SCS}(x_2) \geq 0$ and if $U_2^{CS}(x_2) - U_2^{SCS}(x_2) < 0$ then we also have $U_2'^{CS}(x_2) - U_2'^{SCS}(x_2) < 0$. In other words, the knowledge that bidder 2 is the high signal bidder does not change his preferred strategy, as the difference in the two expected utilities is the same as before, but multiplied by a constant (the term on the left in equation (16)).

(Bidder 1) By not being recommended to start the auction, bidder 1 knows he holds the lowest signal. Hence, assuming bidder 2 obeys the mediator's recommendations, increasing the bid by the least amount possible (straightforward bidding) weakly dominates all other strategies, because bidder 1 knows he will lose the auction and is indifferent between submitting the minimum bid or any other bid below his bidding limit. Also, disobeying the mediator and starting the auction is weakly dominated because bidder 1 knows he will lose the auction irrespective of who starts the auction (note that the 'no ties' assumption is important for this to hold: if ties were allowed, then bidder 1 could expect to win the auction if he started it even though he held the lowest signal).

If both bidders prefer to bid the minimum amount possible at every stage (other than the first one), bidder 1 should not bid more than $V(x_1, x_1)$ given that 2 does not bid more than $V(x_2, x_2)$. If he does, he risks winning the auction and losing money (if he wins the auction, pays a price at least equal to $V(x_2, x_2)$ and loses money because $V(x_1, x_2) < V(x_2, x_2)$). Given that bidder 1's bidding limit is $V(x_1, x_1)$, bidder 2 has no incentives to use a lower or higher bidding limit than $V(x_2, x_2)$. A lower bidding limit would decrease his overall probability of winning the auction without affecting the payoff in case he did win, and a higher bidding limit would not even increase his probability of winning the auction.

(ii) Truthful reports

(Bidder 2) If $x_2 > x_1$, bidder 2 does not want to report $x_2^R < x_2$: by doing so, he risks reporting a signal lower than x_1 and hence not being recognized by the mediator as the high signal bidder (and consequently not being recommended to start the auction, which makes him lose the advantage of holding the highest signal). Bidder 2 does not report $x_2^R > x_2$ because if $x_2 > x_1$, there is no further advantage to be gained by reporting a higher signal. Thus, bidder 2 reports truthfully.

(Bidder 1) If $x_2 > x_1$, bidder 1 does not gain from reporting $x_1^R > x_1$. Such a report could make the mediator believe that bidder 1 held the highest signal, and hence recommend him to start the auction. Because no ties are allowed, after the bidding has reached $V(x_1, x_1)$, bidder 1's bidding limit, bidder 1 prefers to lose the auction and hence would not gain from a dishonest report. Reporting a signal $x_1^R < x_1$ does not benefit him either, because $x_2 > x_1$ and there is no advantage in reporting a lower signal. Thus, bidder 1 reports truthfully to the mediator.

(iii) Communication equilibrium

Hence, the proposed mechanism is a communication equilibrium when there are no ties, because both bidders have incentives to report their signals truthfully to the mediator and to obey its recommendations. ■

Notice that in this (communication) equilibrium, the choice of the initial bid favors the high signal bidder, in the manner of Avery (1998). The high signal bidder will choose the initial bid which favors him most in the subsequent stages, leading to the highest possible expected utility. Also note also that this result holds in the absence of a mediator. The main role of the mediator is to receive the signal reports and recommend the high signal bidder to start the auction with a bid which depends on the expected utility of the various bidding paths available. If the signal ranking is common knowledge, i.e. if both bidders know who holds the highest signal (but not the particular signal realizations), a mediator is not necessary for the equilibrium in Proposition 1 to hold:

Proposition 2 *Assuming there are no ties and the signal ranking is common knowledge, the equilibrium strategies contained in Proposition 1 are equilibrium strategies of the auction game.*

Proof. If the signal ranking is common knowledge, i.e. if after receiving the signals it becomes common knowledge whether $X_1 > X_2$ or $X_1 < X_2$, the role of the mediator described in Proposition 1 can be performed by the bidders themselves.

The bidder holding the highest signal (let us assume, as in Proposition 1, that $x_2 > x_1$) can choose the strategy (closest strategy or second closest strategy) which gives him a higher expected utility. This comparison is done when account is taken of the signal ranking (see equation (16)). Bidder 2 prefers to start the auction with a bid which depends on whether n_2 is odd or even and on the sign of $U_2'^{CS}(x_2) - U_2'^{SCS}(x_2)$, in the same way as described in Proposition 1. That strategy (closest or second closest strategy) is a best response to bidder 1's strategy of bidding straightforwardly and not starting the auction.

In turn, the bidder holding the lowest signal weakly prefers not to start the auction and bid straightforwardly given bidder 2's strategy.

Hence, the equilibrium strategies contained in Proposition 1 are an equilibrium of the auction game provided no ties are allowed and the signal ranking is common knowledge. ■

Note that the knowledge of the signal ranking is, in effect, the introduction into the auction model of noisy and private information about an opponent's signal, as suggested by Fang and Morris (2006). Each bidder knows whether he holds the highest (or lowest) signal and this allows him to eliminate the possibility that his opponent has a higher (or lower) signal than his. For instance, if it became common knowledge that $X_2 > X_1$ (i.e. bidder 2 holds the highest signal), bidder 1 now knows that $X_2 > x_1$. This is a noisy signal of bidder 2's signal realization, because bidder 1 cannot infer with certainty bidder 2's private signal. Similarly, bidder 2 knows that $X_1 < x_2$, which is also a noisy signal about bidder 1's signal realization. These two noisy signals are clearly private: although the signal ranking is common knowledge, each individual noisy signal depends on that bidder's private signal realization. This is similar to Fang and Morris' (2006) setup, although their assumption was that the noisy signals were point estimates, whereas the noisy signals in our model take the form of (restricted) ranges for an opponent's signal realization.

4 An example

In order to illustrate the communication equilibrium, let us consider a specific example. Let us assume that the signals are independently and uniformly distributed in the interval $[0,1]$. Let us also assume that the bid levels are $A = \left\{0, \frac{1}{4}, \frac{1}{2}; \frac{3}{4}; 1\right\}$, i.e. the difference between every two bid levels is constant and equal to $\frac{1}{4}$.

Note that, in this case, equation (9) becomes:

$$U_i^{CS}(x_i) - U_i^{SCS}(x_i) = \begin{cases} z\left(a_{n_i-2}, a_{n_i-1}\right)\left(a_{n_i-1} - a_{n_i-2}\right), \text{ if } n_i \text{ is even} \\ 0, \text{ if } n_i \text{ is odd} \end{cases} \tag{17}$$

For the uniform signal distribution and for equally distanced discrete bid levels, the closest strategy is always (weakly) preferred to the second closest strategy. Assuming that bidder 2 holds the highest signal ($x_2 > x_1$) and reports truthfully to the mediator, and also assuming there are no ties, because $U_2^{CS}(x_2) - U_2^{SCS}(x_2) \geq 0$, the mediator recommends bidder 2 is to start the auction with a bid of a_0 if n_2 is odd and with a bid of a_1 if n_2 is even (and whether n_2 is odd or even depends on bidder 2's particular signal realization), as described in Proposition 1. After this initial bid, the mediator will recommend that bidder 2 increase the bid by the minimum amount possible up to s_2^H (closest strategy). After receiving the recommendation to start the auction, bidder 2 knows that he holds the highest signal and hence the difference between the closest and second closest strategy now becomes:

$$U_2'^{CS}(x_2) - U_2'^{SCS}(x_2) = \begin{cases} \frac{1}{G_{X_1|X_2}(x_2|X_2=x_2)} z\left(a_{n_i-2}, a_{n_i-1}\right)\left(a_{n_i-1} - a_{n_i-2}\right), \text{ if } n_i \text{ is even} \\ 0, \text{ if } n_i \text{ is odd} \end{cases} \tag{18}$$

Thus, the closest strategy continues to (weakly) dominate the second closest strategy even after bidder 2 realizes that he holds the highest signal. Hence, disobeying the mediator would certainly give him a lower expected payoff and he would have no incentives to do so.

Bidder 1 will not be recommended to start the auction and will be recommended to bid the minimum amount possible whenever it is his turn to bid, up to s_1^H. Given that the mediator only makes this recommendation when there are no ties, bidder 1 knows that he will lose the auction because he is the low signal bidder; hence, his expected utility is 0 and disobeying the mediator would not give him a higher utility.

5 Conclusion

This paper has proposed a model of a common value English auction where bidding is discrete. We obtained a communication equilibrium which requires a coordination mechanism or mediator. In this communication equilibrium, bidders have incentives to report their signals truthfully to the mediator and obey his recommendations. The mediator's recommendations depend on the signal realizations of both bidders, but always encompass a recommendation to the high signal bidder to start the auction in such as way as to choose the bidding path which favors him the most. This equilibrium can be played without such a mediator if the signal ranking is common knowledge. In reality, the mediator's role is solely that of receiving the signal reports and suggesting different courses of action to the high and low signal bidder.

Our model combines two recent departures from the standard modeling of English auctions. Firstly, we model the English auction with discrete bidding instead of modeling it as the Japanese English auction with continuous bidding (Milgrom and Weber (1982)). Secondly, we incorporate the possibility that, before the auction starts, bidders know more than simply their private signal realizations. In particular, in the equilibrium we have obtained, bidders must know the signal ranking before the auction starts, which in effect is a noisy signal about the opponent's private information (Fang and Morris (2006)). The results obtained suggest that bidding in such English auctions can be largely driven by the strategic concerns mentioned in Isaac et al. (2007), i.e. bidders choose their bids to prevent their opponents' from doing so.

In the future, it would be interesting to see whether our result holds in a model with known and predefined discrete bid increments, such as that of Isaac et al. (2007). Such a result would be important, as it could be applicable to bidding behavior in spectrum auctions. It would also be interesting to further investigate whether bidding behavior is materially changed when bidders know more than the signal ranking, e.g. when they know (with some imprecision), for instance, the interval to which their opponent's bidding limit belong. Such an assumption could well yield other equilibrium strategies driven by strategic concerns.

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